Geometry & Differential Geometry

1) Representations of S_n

- Recall that rep theory of finite groups tells us that:
  - reps of S_n are semisimple
  - # irreps of S_n = # conjugacy classes = # partitions of n.
- Combinatorics tells us that

$$\{\text{irreps of } S_n\} \longleftrightarrow \{\text{partitions of } n\} \longleftrightarrow \left\{\begin{matrix}\text{standard Young}\\ \text{tableau}\end{matrix}\right\}$$

  - These can be constructed by the Specht modules
  - While conjugacy classes are always in bijection with irreps, it's rare to have such an explicit bijection.

2) As a Weyl group

- Recall that SL_n = $n \times n$ matrices in $\mathbb{C}$ with $\det = 1$
- The Lie algebra $\mathfrak{sl}_n :=$ tangent space at $1 \in SL_n$.
  - Just a vector space!
  - This can be computed by

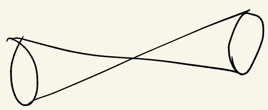
$$\ker(SL_n(\mathbb{C}[\varepsilon]/\varepsilon^2) \to SL_n(\mathbb{C}))$$
$$= \{ Id + \varepsilon \cdot M \mid M \in Mat_{n \times n}(\mathbb{C}),\ \det(Id + \varepsilon \cdot M) = 1\}$$
$$= \{ M \in Mat_{n\times n}(\mathbb{C}) \mid \prod_{i=1}^{n} (1 + \varepsilon \cdot m_{ii}) = 1 \Longleftrightarrow \sum_{i=1}^{n} m_{ii} = 0 \}.$$

- The Lie algebra $\mathfrak{sl}_n$ has a natural action of SL_n, by conjugation.
- The Weyl group of $\mathfrak{sl}_n$ is S_n.
  - So S_n appears "geometrically".

$\underline{Q}$ Can we construct its rep's geometrically?

3) Nilpotent cone

- Def: $x \in \mathfrak{sl}_n$ is <u>nilpotent</u> if it acts nilpotently on every f.d. $\mathfrak{gl}_n$-module.
  - This is equivalent to being a nilpotent matrix.
- Def: the <u>nilpotent cone</u> $\mathcal{N} \subseteq \mathfrak{sl}_n$ is the subvariety of nilpotent elements.
  - <u>Ex</u>: for $\mathfrak{sl}_2$, $\mathcal{N} = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{sl}_2 \mid ad-bc=0 \}$



  - This is always a singular variety, with one singular point at 0.
- The SL_n-orbits are just conjugacy classes of nilpotent matrices. By Jordan form, this is just partitions of n.

$$\left\{ \begin{matrix} SL_n\text{-orbits} \\ \text{in } \mathcal{N} \end{matrix} \right\} \longleftrightarrow \{ \text{partitions of } n \}.$$

- Where have we seen this before?

4) Some structure theory

Fix: $-\ G = SL_n$.

$-\ T$: diagonal matrices $\subseteq SL_n$ $\begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$

Borel $-\ B$: upper triangular $\subseteq SL_n$ $\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$

$-\ \mathfrak{h} = \mathrm{Lie}(T) \subset \mathfrak{sl}_n$: $\begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$

$-\ \mathfrak{b} = \mathrm{Lie}(B) \subset \mathfrak{sl}_n$: $\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$

<u>Fact</u>

1) All Borels (B or $\mathfrak{b}$) are conjugate by SL_n

2) Stabilizer is B

<u>Def</u> Flag variety $\mathcal{F} :=$ moduli space of all Borels.

$\mathcal{F} \xrightarrow{\sim} G/B$ after fixing a Borel B.

<u>Remark</u>

A choice of Borel in $SL_n \longleftrightarrow$ full flag $0 \subseteq \mathbb{C} \subseteq \cdots \subseteq \mathbb{C}^n$.

4) Springer resolution

- Consider the incidence variety

$$\tilde{\mathcal{N}} := \{ (x, \mathfrak{b}) \in \mathcal{N} \times G/B \mid x \in \mathfrak{b} \}.$$

  - This is a smooth variety.
  - Note that $\mathcal{N} \cap \mathfrak{b} = \mathfrak{n}$, strictly upper triangular.
  - So $\tilde{\mathcal{N}} \simeq G \times^B \mathfrak{n}$.
  - But $\mathfrak{n} \simeq \mathfrak{b}^\perp$ after using Killing form $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$.
  - So $\tilde{\mathcal{N}} \simeq G \times^B \mathfrak{b}^\perp \simeq T^* G/B$. Cor: $\tilde{\mathcal{N}}$ is smooth.

    (cotangent spaces varies with point)
- $\tilde{\mathcal{N}}$ comes with two natural projections:

$$\mathcal{N} \xleftarrow{\mu} \tilde{\mathcal{N}} \xrightarrow{\pi} G/B.$$

  - The second map $\pi : \tilde{\mathcal{N}} \to G/B$ is precisely the natural map $\pi : T^* G/B \to G/B$.
  - The first map is the moment map $T^* G/B \to \mathfrak{g}^*$ arising from the G-action on $T^* G/B$.
- The map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is called the <u>Springer resolution</u>.

5) Springer fibers

- Recall that

$$\{\text{irreps of } S_n\} \longleftrightarrow \{\text{partitions of } n\} \longleftrightarrow \left\{\begin{matrix} SL_n\text{-orbits} \\ \text{in } \mathcal{N} \end{matrix}\right\}$$

- Note that $\mu: \tilde{\mathcal{N}} \longrightarrow \mathcal{N}$ is a SL_n-equiv. map.

  So the fibers depend only on SL_n-orbits in $\mathcal{N}$.

- Get a different fiber over each SL_n-orbit in $\mathcal{N}$. These are called Springer fibers.
  - Concretely, for $e \in \mathcal{N}$, $\mu^{-1}(e) = \{\text{Borels } \mathfrak{b} \text{ containing } e\}$
    $= \{\text{full flags stabilized by } e\}$.

Punchline

$$\{SL_n\text{-orbits in } \mathcal{N}\} \xrightarrow[\sim]{\mathbb{O} \longmapsto H^{BM}_{top}(\mathbb{O})} \{\text{irreps of } S_n\}$$

Remark: This is a truly "geometric" action since the S_n-action is induced by Borel-Moore convolution by another variety: Steinberg variety, whose (top) BM-homology is $\mathbb{C}[S_n]$.

6) Extensions

- Can construct $\mathfrak{sl}_n$-irreps in a similar fashion.
  - have to use partial flag varieties (flag length $= n$)
    - Can construct in much the same way... but need to fix the ambient space $\mathbb{C}^d$.
    - once again, algebra $= H^{BM}_{top}$ ("Steinberg"), irreps $= H^{BM}_{top}$ ("Springer fibers")
    - this construction generates only a finite-dim algebra via H^{BM}_{top} ("Steinberg"), but $\mathcal{U}(\mathfrak{sl}_n)$ is infinite-dim. This can't get us all $\mathfrak{sl}_n$-irreps!
    - Answer is Schur-Weyl duality. It turns out that the constructed algebra is $\mathcal{U}(\mathfrak{sl}_n)/\mathrm{Ann}((\mathbb{C}^n)^{\otimes d})$.
    - As $d \to \infty$, we get <u>all</u> $\mathfrak{sl}_n$-irreps.
- There is a somewhat unrelated idea, the Lustig-Vogan bijection,

$$\{\text{irreps of } G\} \longleftrightarrow \left\{ (\mathbb{O}, \mathcal{E}) \;\; \begin{array}{l} \mathbb{O}\text{- nilpotent orbit} \\ \mathcal{E}\text{- irrep of stabilizer } G_x,\ x \in \mathbb{O} \end{array} \right\}$$

which seems related to Springer correspondence (but as far as I can, is not)

  - The bijection is as follows
    - $\mathcal{E} \rightsquigarrow G \times^{G_x} \mathcal{E}$ gives $\tilde{\mathcal{E}}$, an irred. G-equiv coh. sheaf on $\mathbb{O}$
    - $(\mathbb{O}, \mathcal{E}) \mapsto$ minimal $\lambda \in \Lambda^+$ s.t. $\exists$ G-equiv coh sheaf $\mathcal{F}$ on $\mathcal{N}$, s.t. $\operatorname{supp} \mathcal{F} \subseteq \overline{\mathbb{O}}$, and $\mathcal{F}|_{\mathbb{O}} = \tilde{\mathcal{E}}$
      - alternatively, $[\mathcal{F}] \in K^G(\mathcal{N}) \simeq \mathbb{C}[\Lambda]^W$, $[\mathcal{F}] = \lambda +$ lower order terms
- Finally, I'll remark that the strategy here can be applied more generally to construct irreps of other important algebras, e.g.
  - Kac-Moody Lie algebras (Nakajima quiver varieties)
  - Heisenberg algebra (Hilbert schemes)
  - Quantized universal enveloping alg of (affinization) $\mathfrak{g}_Q$ (equivariant K-theory)